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Period Doubling Solutions in the Duffing Oscillator: A Galerkin Approach

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The period doubling solutions for the attractor introduced by Huberman and Crutchfield are studied in the relevant range for the forcing frequency by a numerical method based on the computation of Galerkin Fourier approximations with very high accuracy. For answering in an affirmative manner the questions of the existence of an exact period doubled solution, the error estimate and the transition from a stable to an unstable solution, very precise solutions combined with periodic series representations are required. It is shown that the algorithm which uses Galerkin Fourier approximations of high order is a very efficient one.

1. INTRODUCTION

In the last decennium the study of period doubling as one of the routes to chaotic behaviour has aroused much attention by applied scientists and engineers [1–5]. The related topic of strange attractors in deterministic systems has grown to be one of the main topics in the study of non-linear phenomena in dynamical systems. Period doubling in non-linear oscillators of the Duffing type has been intensely studied [5–8]. Numerical methods on the computation of periodic solutions and on period doubling bifurcations are described in [9–11].

Huberman and Crutchfield [7] investigate the behaviour of particles in anharmonic potentials moving in an external periodic field. The governing equation of motion for the charge written in dimensionless form reads

$$\frac{d^2x}{d\tau^2} + c\frac{dx}{d\tau} + x - 4x^3 = g\cos\Omega\tau, \tag{1.1}$$

where g is the amplitude of the external periodic field, Ω is the frequency, and c

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represents the damping coefficient. Equation (1.1) was solved by using a Systron–Donner analog computer. Choosing $c = 0.4$ and $g = 0.115$ the amplitude-frequency response curve exhibits the well-known pattern characterized by a bending to the left [6], since the Duffing oscillator has a non-linear characteristic of the soft spring type.

If one decreases Ω starting with a limit cycle of period $2\pi/\Omega$ from $\Omega = 1.5$, a set of cascading period doubling bifurcations starts taking place at $\Omega \approx 0.56$ resulting finally in a strange attractor in phase space which is attained at the threshold value $\Omega = 0.5567$. In addition the usual hysteresis loop was found. These results of Huberman and Crutchfield have been confirmed by Räty *et al.* [8] if one takes into account that analog computer results are only accurate to a few percent. By a numerical simulation based on a Runge–Kutta–Verner integration method of (1.1), the period doubling bifurcations of the intersection in the phase plane with the negative x-axis have been represented. The symmetry of the harmonic solution is broken at $\Omega \approx 0.535$. Period doubling bifurcations arise at $\Omega \approx 0.53$ and the transition to chaos occurs at $\Omega \approx 0.528$.

In [12] Van Dooren investigated the transition from regular to chaotic behaviour in the Duffing oscillator, especially the determination of the harmonic solution having the fundamental period and its related stability analysis.

The aim of the present work is to study in a complete manner the period doubling solutions of the Duffing oscillator with a soft characteristic as considered in [7, 8] in the relevant interval of interest for the forcing frequency ranging from $\Omega = 0.528$ to $\Omega = 0.53$. Independently of the methods used in [7, 8] the Galerkin method [13, 14] will be applied here. The Galerkin approximations representing period doubling will be computed with very high precision. The non-linear determining equations for the unknown coefficients in the series representations are solved by applying Newton's iterative method. However, the Galerkin method has the advantage to yield a complete criterion. Once an approximate periodic solution has been computed, the method allows us to answer the fundamental question whether there exists an exact periodic solution in a small neighbourhood of a computed Galerkin approximation. In the affirmative case it also yields an error estimate of this Galerkin approximation. In addition the stability of the solution may be investigated. This is performed by computing the eigenvalues of a certain matrix related to the first variational equations of the original system in conjunction with the computed Galerkin approximations.

Obviously the choice of the use of the considered Galerkin Fourier approximations as approximate periodic solutions is a natural choice which inherently fulfils the requirement of periodicity. However, it has been pointed out that if such Galerkin approximations of low order are used, it is not possible to answer in an affirmative manner the mathematical questions on the existence problem and the error estimation. Therefore, for answering the mathematical questions, very accurate solutions in conjunction with inherent periodic series representations are needed. It will be shown in this work that precisely these requirements are fulfilled by the considered Galerkin Fourier approximations of high order.

It is to be emphasized that the results derived by the Galerkin method confirm the results found in [8]. However, the results obtained here give additional insight in the aspects of existence, error estimate, and stability, i.e., the mathematical aspects which are precisely lacking in most methods of solution. In a series of papers Van Dooren [15–20] has illustrated that the Galerkin method is a very efficient algorithm in many important fields of applied sciences and engineering. In a forthcoming paper Janssen and Van Dooren will report their results on a numerical study of the Duffing attractor by using very accurate Gauss–Legendre integration methods.

2. Period Doubling Solutions by the Galerkin Method

Let us introduce the new independent variable

$$t = \frac{\Omega}{2}\tau. \tag{2.1}$$

Equation (1.1) is then rewritten as

$$\ddot{x} = X(x, \dot{x}, t), \tag{2.2}$$

with

$$X(x, \dot{x}, t) = -W^2 x - Ex^3 - C\dot{x} + G\cos 2t, \tag{2.3}$$

$$W = \frac{2}{\Omega}, \qquad E = -\frac{16}{\Omega^2}, \qquad C = \frac{2}{\Omega}c, \qquad G = \frac{4g}{\Omega^2}. \tag{2.4}$$

A dot now indicates differentiation with respect to t. Note that the coefficient E of the non-linearity in the relevant Ω region (0.528, 0.53) is considerably large and that the period of the forced term has been transformed to the value π. The phenomenon of period doubling oscillations is thus represented by 2π periodic solutions. Therefore, following Urabe's approach [14] applied to the Galerkin method [13], one looks for an approximate periodic solution to (2.2) which is represented by a trigonometric polynomial of order m of the form

$$x_m(t) = a_0 + \sum_{n=1}^{m} (a_{2n-1}\sin nt + a_{2n}\cos nt), \tag{2.5}$$

with unknown coefficients $a_0, a_1, ..., a_{2m}$. These unknowns are determined by applying a balance procedure to the equation

$$\ddot{x}_m(t) = X_m[x_m(t), \dot{x}_m(t), t], \tag{2.6}$$

where $X_m[x_m(t), \dot{x}_m(t), t]$ represents the Fourier series of $X[x_m(t), \dot{x}_m(t), t]$ truncated at the harmonics of order m:

$$X_m[x_m(t), \dot{x}_m(t), t] = A_0 + \sum_{n=1}^{m} (A_{2n-1} \sin nt + A_{2n} \cos nt), \tag{2.7}$$

A_j being the Fourier coefficients.

Let us add a few comments to clarify the Galerkin procedure. The left-hand side in (2.6) is the trigonometric polynomial of order m:

$$\ddot{x}_m(t) = -\sum_{n=1}^{m} n^2(a_{2n-1} \sin nt + a_{2n} \cos nt). \tag{2.8}$$

For a general function $X(x, \dot{x}, t)$ the Fourier series of $X[x_m(t), \dot{x}_m(t), t]$ in which $\dot{x}_m(t)$ is represented by

$$\dot{x}_m(t) = \sum_{n=1}^{m} n(-a_{2n} \sin nt + a_{2n-1} \cos nt), \tag{2.9}$$

has an infinite number of terms in its development. It is to be emphasized that in the Galerkin approach, while considering a trigonometric polynomial of order m for the approximate periodic solution, the harmonics of order higher than m, which generally occur, are neglected in the Fourier series development of $X[x_m(t), \dot{x}_m(t), t]$. The Fourier series coefficients are given by

$$\begin{aligned} A_0 &= \frac{1}{2\pi} \int_0^{2\pi} X[x_m(t), \dot{x}_m(t), t]\, dt, \\ A_{2n-1} &= \frac{1}{\pi} \int_0^{2\pi} X[x_m(t), \dot{x}_m(t), t] \sin nt\, dt, \\ A_{2n} &= \frac{1}{\pi} \int_0^{2\pi} X[x_m(t), \dot{x}_m(t), t] \cos nt\, dt, \end{aligned} \tag{2.10}$$

with $n = 1, 2, ..., m$. In general, these Fourier coefficients are non-linear functions in the $2m+1$ unknowns a_j represented in the sequel by the set $a \equiv (a_0, a_1, ..., a_{2m})$. The truncation of the Fourier series of $X[x_m(t), \dot{x}_m(t), t]$ at the harmonics of order m in (2.6), then allows the determination of the $2m+1$ unknowns a_j by equating the coefficients of the set of the $2m+1$ basic functions 1, $\sin nt$, $\cos nt$ with $n = 1, 2, ..., m$ in (2.6). Hence the following determining equations for the unknown coefficients are obtained:

$$\begin{aligned} F_0(a) &\equiv A_0(a) = 0, \\ F_{2n-1}(a) &\equiv A_{2n-1}(a) + n^2 a_{2n-1} = 0, \\ F_{2n}(a) &\equiv A_{2n}(a) + n^2 a_{2n} = 0, \end{aligned} \tag{2.11}$$

with $n = 1, 2, ..., m$. The trigonometric polynomial (2.5) found by this procedure is called a Galerkin approximation of order m.

The set of the $2m+1$ non-linear equations (2.11) in the $2m+1$ unknowns a_j can then be solved, e.g., by applying the generalized Newton method whereby one linearizes (2.11) near an approximate solution $\bar{a}$, yielding

$$F_i(\bar{a}) + \sum_{j=0}^{2m} \frac{\partial F_i}{\partial a_j}(\bar{a})(a_j - \bar{a}_j) = 0, \tag{2.12}$$

with $i = 0, 1, ..., 2m$. This procedure needs the computation of F_i and its partial derivatives $\partial F_i/\partial a_j$ when the values of an approximate solution $\bar{a}$ to (2.11) are given. This is provided for by a numerical approximation method described in [14] which is based on the approximation formulae for the computation of the Fourier coefficients A_j in (2.10),

$$\begin{aligned} A_0 &= \frac{1}{2N} \sum_{k=1}^{2N} X[x_m(t_k), \dot{x}_m(t_k), t_k], \\ A_{2n-1} &= \frac{1}{N} \sum_{k=1}^{2N} X[x_m(t_k), \dot{x}_m(t_k), t_k] \sin nt_k, \\ A_{2n} &= \frac{1}{N} \sum_{k=1}^{2N} X[x_m(t_k), \dot{x}_m(t_k), t_k] \cos nt_k, \end{aligned} \tag{2.13}$$

with $n = 1, 2, ..., m$; $N > m$ and $t_k = [(2k-1)/2N]\pi$.

Let us explain how the starting values needed in applying Newton's iterative method were obtained. In [12] Van Dooren studied the harmonic solution to Eq. (1.1) having the fundamental period. The accurate determination of such a harmonic solution depends on the initial approximation which was found to be [12]

$$\bar{x}(\tau) = \frac{c\Omega A^2}{g} \sin \Omega\tau + (1 - \Omega^2 - 3A^2)\frac{A^2}{g} \cos \Omega\tau, \tag{2.14}$$

where A satisfies

$$[(1 - \Omega^2 - 3A^2)^2 + c^2\Omega^2]A^2 = g^2. \tag{2.15}$$

The starting values, needed here in applying Newton's iterative method for obtaining the period doubled solutions, were derived from the Galerkin approximation of order $m = 2$. Thus, taking into account that $t = (\Omega/2)\tau$ according to the transformation (2.1), the initial approximation here is of the form

$$\bar{x}(t) = \bar{a}_0 + \bar{a}_1 \sin t + \bar{a}_2 \cos t + \bar{a}_3 \sin 2t + \bar{a}_4 \cos 2t. \tag{2.16}$$

The five determining equations for the five unknowns were solved by an approximation technique whereby a_0, a_1, and a_2 are small. The result is that $\bar{a}_3$ and

$\bar{a}_4$ are given by the expressions of the coefficients of $\sin \Omega\tau$ and $\cos \Omega\tau$ in (2.14). In addition $\bar{a}_0$, $\bar{a}_1$, and $\bar{a}_2$ are obtained from certain analytical expressions depending upon $\bar{a}_3$ and $\bar{a}_4$.

For those values of Ω where periodic solutions with the fundamental period and the doubled period coexist, the solution depends on the initial approximation. Starting values are taken from (2.14) for computing the harmonic solution and from (2.16) for obtaining the subharmonic solution with the period doubled.

It has been pointed out that additional periodic solutions in the period doubling cascade can be derived in an analogous manner as described here. The study of, e.g., the transition from $2T$ to $4T$ proceeds analogously as the transition from T to $2T$, where T represents the fundamental period.

Performing the computations on a CDC CYBER 170/750 computer, Galerkin approximations have been calculated for the values $c = 0.4$ and $g = 0.115$ taken in

TABLE I.

The Galerkin Approximation of Various Order m for $\Omega = 0.529$

m	15	20	25
a_0	-0.0443744484	-0.0443730341	-0.0443727620
a_1	0.0099194773	0.0097723337	0.0097728507
a_2	0.0097799938	0.0096278057	0.0096283157
a_3	0.5026113565	0.5026172283	0.5026172741
a_4	-0.0042860123	-0.0042623811	-0.0042624049
a_5	-0.0055030463	-0.0054220799	-0.0054223821
a_6	0.0026017614	0.0025582279	0.0025583739
a_7	0.0083363630	0.0083400295	0.0083399786
a_8	-0.0367240566	-0.0367259990	-0.0367257708
a_9	0.0028143694	0.0027720007	0.0027721471
a_{10}	0.0044335803	0.0043657115	0.0043659414
a_{11}	0.0381813869	0.0381861250	0.0381861558
a_{12}	0.0067954257	0.0068033621	0.0068033641
a_{13}	-0.0006720500	-0.0006625511	-0.0006625962
a_{14}	0.0001530147	0.0001496746	0.0001496863
a_{15}	0.0022033545	0.0022064715	0.0022064585
a_{16}	-0.0063355454	-0.0063386318	-0.0063385914
a_{17}	0.0003883602	0.0003823067	0.0003823274
a_{18}	0.0008663158	0.0008531867	0.0008532361
a_{19}	0.0029727313	0.0029751255	0.0029751357
a_{20}	0.0007503386	0.0007525750	0.0007525776
a_{21}	-0.0000069882	-0.0000072211	-0.0000072280
a_{22}	-0.0000139921	-0.0000138646	-0.0000138641
a_{23}	0.0003667563	0.0003710591	0.0003710588
a_{24}	-0.0007865657	-0.0007912413	-0.0007912362

TABLE I—*Continued*

m	15	20	25
a_{25}	0.0000366582	0.0000360401	0.0000360427
a_{26}	0.0001216124	0.0001202470	0.0001202674
a_{27}	0.0002037595	0.0002052135	0.0002052221
a_{28}	0.0000586210	0.0000598510	0.0000598520
a_{29}	0.0000115178	0.0000112214	0.0000112143
a_{30}	-0.0000034207	-0.0000034198	-0.0000034190
a_{31}		0.0000501009	0.0000501067
a_{32}		-0.0000842129	-0.0000842149
a_{33}		0.0000024603	0.0000024572
a_{34}		0.0000138779	0.0000139221
a_{35}		0.0000119317	0.0000119508
a_{36}		0.0000024517	0.0000024517
a_{37}		0.0000026643	0.0000026653
a_{38}		-0.0000002598	-0.0000002599
a_{39}		0.0000058140	0.0000058309
a_{40}		-0.0000080592	-0.0000080704
a_{41}			0.0000001052
a_{42}			0.0000013717
a_{43}			0.0000004898
a_{44}			-0.0000002316
a_{45}			0.0000004238
a_{46}			0.0000000171
a_{47}			0.0000006018
a_{48}			-0.0000007129
a_{49}			0.0000000013
a_{50}			0.0000001138

[7, 8] for various variations of m and of the frequency in the relevant region ranging from $\Omega = 0.528$ to $\Omega = 0.53$. Table I gives the coefficients of the Galerkin approximation with $\Omega = 0.529$ for three variations of the order m, i.e., $m = 15$, $m = 20$, and $m = 25$ taking $N = 3m$. Numerical convergence with respect to the coefficients is noticed when the order m is increased. Table II represents the Galerkin coefficients with $m = 25$ for various values of Ω. These coefficients were obtained after four iterations with a required precision of 10^{-10} between two consecutive iterations in order to stop Newton's iterative method.

In conclusion it is mentioned that the computation of period doubling solutions can be performed by applying the Galerkin method in combination with the Newton algorithm. Although the coefficient E of the non-linear characteristic is large ($E = -57.175$ for $\Omega = 0.529$) it can be concluded that the method is a very efficient one if one looks for approximate period doubling solutions having a very high precision.

TABLE II

The Galerkin Approximation of Order $m = 25$ for Various Ω

Ω	0.5285	0.5295	0.5299
a_0	-0.0445125644	-0.0442353239	-0.0441271363
a_1	0.0119933086	0.0068884194	0.0030327080
a_2	0.0117204951	0.0068420917	0.0030321083
a_3	0.5027265426	0.5025082481	0.5024212432
a_4	-0.0046112549	-0.0039144188	-0.0036367189
a_5	-0.0066476223	-0.0038259113	-0.0016858014
a_6	0.0030891236	0.0018326429	0.0008172947
a_7	0.0083399292	0.0083418963	0.0083448102
a_8	-0.0368581017	-0.0365954772	-0.0364927569
a_9	0.0034125810	0.0019478362	0.0008553903
a_{10}	0.0053317304	0.0030926004	0.0013669958
a_{11}	0.0382086512	0.0381638871	0.0381462588
a_{12}	0.0067111494	0.0068949383	0.0069677251
a_{13}	-0.0008123730	-0.0004674675	-0.0002059587
a_{14}	0.0001730041	0.0001117075	0.0000513835
a_{15}	0.0022006925	0.0022128216	0.0022183531
a_{16}	-0.0063598111	-0.0063176693	-0.0063011543
a_{17}	0.0004723020	0.0002676903	0.0001172190
a_{18}	0.0010436067	0.0006034425	0.0002664003
a_{19}	0.0029744609	0.0029758648	0.0029764925
a_{20}	0.0007353432	0.0007696108	0.0007830918
a_{21}	-0.0000086176	-0.0000052356	-0.0000023532
a_{22}	-0.0000188015	-0.0000087354	-0.0000034775
a_{23}	0.0003683044	0.0003739152	0.0003762762
a_{24}	-0.0007927408	-0.0007897432	-0.0007885584

3. Existence, Error, and Stability Analysis

By a theorem and a numerical approximation method due to Urabe [14] one can verify the existence of an exact isolated periodic solution $\hat{x}(t)$ in a small neighbourhood with radius d of a numerically computed Galerkin approximation $\bar{x}(t)$ and in the affirmative case we can yield an error estimate e correxponding to this Galerkin approximation:

$$\max_{0 \leqslant t \leqslant 2\pi} \|\hat{x}(t) - \bar{x}(t)\| \leqslant e \qquad \text{with} \quad e = \frac{Mr}{1 - K}.$$

The application of this theorem requires the computation of the error estimate r corresponding to the differential equation (2.2) for the computed Galerkin

TABLE II—*Continued*

Ω	0.5285	0.5295	0.5299
a_{25}	0.0000447847	0.0000250861	0.0000109319
a_{26}	0.0001471672	0.0000850186	0.0000375188
a_{27}	0.0002048141	0.0002056439	0.0002059919
a_{28}	0.0000573607	0.0000623028	0.0000642341
a_{29}	0.0000130264	0.0000078684	0.0000034517
a_{30}	-0.0000044700	-0.0000022505	-0.0000009341
a_{31}	0.0000493878	0.0000508381	0.0000514326
a_{32}	-0.0000841806	-0.0000842452	-0.0000842667
a_{33}	0.0000030954	0.0000016860	0.0000007261
a_{34}	0.0000170264	0.0000098470	0.0000043473
a_{35}	0.0000119067	0.0000119984	0.0000120391
a_{36}	0.0000021706	0.0000027267	0.0000029422
a_{37}	0.0000032809	0.0000018729	0.0000008226
a_{38}	-0.0000003592	-0.0000001598	-0.0000000620
a_{39}	0.0000056936	0.0000059694	0.0000060811
a_{40}	-0.0000080459	-0.0000080937	-0.0000081116
a_{41}	0.0000001394	0.0000000682	0.0000000280
a_{42}	0.0000016746	0.0000009718	0.0000004296
a_{43}	0.0000004890	0.0000004913	0.0000004931
a_{44}	-0.0000002537	-0.0000002102	-0.0000001936
a_{45}	0.0000005211	0.0000002981	0.0000001310
a_{46}	0.0000000152	0.0000000154	0.0000000080
a_{47}	0.0000005806	0.0000006232	0.0000006403
a_{48}	-0.0000007092	-0.0000007164	-0.0000007191
a_{49}	0.0000000028	0.0000000002	-0.0000000001
a_{50}	0.0000001385	0.0000000809	0.0000000359

approximation and the computation of a bound M for the norm of a linear mapping in the space of the continuous 2π-periodic functions associated with a certain Green operator. K is a certain contraction mapping factor. It is also required that all the eigenvalues s_i of a certain matrix $P(2\pi)$ are different from 1. In addition, if all these eigenvalues s_i are in modulus less than 1, then the periodic solution is stable. For further details about Urabe's method the reader is referred to [14].

Table III summarizes the results of the existence and error analysis taking $d=e$ for the Galerkin approximations with $m=40$ for five variations of Ω. In all cases the existence of an exact periodic solution in the neighbourhood with radius d of the relevant Galerkin approximation is guaranteed and the error estimate e is small.

Let us emphasize that it has been pointed out that Galerkin approximations of low order were not sufficiently accurate to answer in an affirmative way the mathematical questions on existence and error estimation. However, the Galerkin

TABLE III

The Existence and Error Analysis with $d = e$

Ω	r	M	K	e
0.52850	5.729×10^{-9}	144.53	0.02825	8.521×10^{-7}
0.52875	5.548×10^{-9}	167.41	0.03683	9.642×10^{-7}
0.52900	5.429×10^{-9}	200.87	0.05243	1.151×10^{-6}
0.52925	5.180×10^{-9}	255.07	0.08279	1.441×10^{-6}
0.52950	4.884×10^{-9}	359.90	0.01705	2.119×10^{-6}

approximations of high order are very efficient for answering those mathematical questions as illustrated here.

Another very interesting feature that can be studied from the stability analysis is the transition from a stable to an unstable period doubled solution. In Table IV one finds the eigenvalues s_1 and s_2 of the matrix $P(2\pi)$ for the period doubled solutions in the relevant frequency domain ranging from a value of Ω slightly below $\Omega = 0.53$, where the creation of a period doubled solution takes place. This creation occurs at the passage in the limit through the value $s = 1$ and its corresponding Ω value has been found to be $\Omega = 0.529995$. When Ω is decreased the solution remains stable until s_2 passes through the value $s = -1$. It has been pointed out that the transition from stability to instability occurs at $\Omega = 0.528608$. At values beyond this transition value the period doubled solutions are unstable.

In conclusion it has been illustrated that the Galerkin method is a very appropriate algorithm for computing accurate period doubled solutions of the Duffing oscillator having a soft characteristic as considered in [7, 8]. In addition this Galerkin method allows us to derive complete information with regard to the mathematical aspects of existence analysis, error bound, and stability behaviour.

TABLE IV

The Stability Analysis

Ω	s_1	s_2	stability
0.52825	-0.0000467903	-1.575168	unstable
0.52850	-0.0000632907	-1.169762	unstable
0.52875	-0.0000956809	-0.777258	stable
0.52900	-0.0001880680	-0.397216	stable
0.52925	-0.0028010458	-0.026790	stable
0.52950	0.0002310536	0.326232	stable
0.52975	0.0001128378	0.671011	stable
0.52990	0.0000870118	0.872514	stable

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